stichting
mathematisch
centrum

AFDELING NUMERIEKE WISKUNDE NW 144/82 DECEMBER
(DEPARTMENT OF NUMERICAL MATHEMATICS)

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# ALGEBRAICALLY EQUIVALENT LINEAR MULTISTEP SOLUTIONS OF VOLTERRA INTEGRAL EQUATIONS AND CERTAIN SYSTEMS OF ODES

Preprint

kruislaan 413 1098 SJ amsterdam

*Printed at the Mathematical Centre, 413 Kruislaan, Amsterdam.*

*The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O.).*

1980 Mathematics subject classification: 65R20

Algebraically equivalent linear multistep solutions of Volterra integral equations and certain systems of ODEs *)

by

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ABSTRACT

It is shown that a special linear multistep method for Volterra integral equations of the second kind with finitely decomposable kernel, is algebraically equivalent with a linear multistep method applied to a certain system of ODEs. Furthermore, a theorem is proved which describes the effect on numerical solutions, generated by a rather general class of linear multistep methods, if the kernel function is replaced by a finitely decomposable kernel.

KEY WORDS & PHRASES: *Numerical analysis, Volterra integral equations of the second kind, linear multistep methods, stability*

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*) This report will be submitted for publication elsewhere.

## 1. INTRODUCTION

A rather general class of linear multistep type methods (*VLM methods*) for solving numerically the Volterra integral equation of the second kind,

(1.1) $$y(t) = g(t) + \int_{t_0}^{t} K(t,s,y(s))ds, \quad t \in I := [t_0,T]$$

consists of the *LM formula*

(1.2a) $$\sum_{i=0}^{k} \{\alpha_i y_{n-i} + \sum_{j=-k}^{k} [\beta_{ij} Y_{n-i}(t_{n+j}) - \gamma_{ij} h K_{n-i}(t_{n+j})]\} = 0,$$

and of a *quadrature rule* for defining the *lagterm* $Y_n(t)$, i.e.

(1.2b) $$Y_n(t) = g(t) + h \sum_{j=0}^{n} w_{nj} K_j(t), \quad K_j(t) := K(t,t_j,y_j).$$

In this VLM method, $y_n$ denotes the numerical approximation to $y(t_n)$, h is the (constant) integration step $t_n - t_{n-1}$, and the parameter matrices $W = (w_{ij})$, $A = (\alpha_i)$, $B = (\beta_{ij})$ and $C = (\gamma_{ij})$ are to be prescribed. These parameters should satisfy $\beta_{ij} = \gamma_{ij} = 0$ for $j < -i$. Since (1.2) is a multistep method, we also need starting values for $y_n$ and $Y_n(t)$, $n = 0 (1) k-1$.

The general VLM method (1.2) was considered in [10], where convergence results and a few stability results were reported. In this paper, we continue this analysis by using the class of *finitely decomposable* kernels as test kernels. It should be remarked that in a few earlier papers (e.g. [3] and [9]), various numerical schemes has already been investigated by restricting the kernel to the class of decomposable kernels. For linear multistep methods, this was done for the conventional direct quadrature (DQ) methods (generated bij (1.2) for $k = 0$, $A = 1$, $B = -1$, and $C = 0$). Here, we consider the general VLM method and we derive a finite terms recurrence relation for the numerical solution. In particular, it will be shown that a special family of VLM methods can be identified with an LM method for a certain system of differential equations. This enables us to get more insight into the numerical behaviour of the VLM solution. Furthermore, a comparison theorem will be proved which describes the effect on the numerical solution of (1.2) if K is replaced by a decomposable approximation.

## 2. FINITELY DECOMPOSABLE KERNELS

It is well know (cf. [7], [6] that (1.1) can be converted into a system of ordinary differential equations (ODEs) if the kernel function K is *finitely decomposable*, that is if

$$(2.1)\qquad K(t,s,y) = \sum_{\mu=1}^{m} g_\mu(t) f_\mu(s,y),$$

or more compactly, in order to simplify the subsequent formulas,

$$(2.1')\qquad K(t,s,y) = < \vec{G}(t), \quad \vec{F}(s,y) >,$$

where <, > denotes the inner product, and $\vec{G}$ and $\vec{F}$ are vectors with components $g_\mu$ and $f_\mu$, $\mu = 1\ (1)\ m$. Introducing the new variable

$$\vec{U}(t) := \int_{t_0}^{t} \vec{F}(s,y(s))ds$$

and using (1.1) we arrive at the system of ODEs

$$(2.2)\qquad \begin{aligned} &\vec{U}'(t) = \vec{F}(t,y(t)) \\ &y(t) = g(t) + < G(t), \quad \vec{U}(t) > \end{aligned}$$

We now try to do a simular conversion of the discrete scheme (1.2). For that purpose we restrict the quadrature rule used in (1.2b) to be class of so-calles $(\rho,\sigma)$-*reducible rules*, that is rules which can be traced back to an LM formula for ODEs (cf. [12] and [15]). The weights of such rules satisfy the relations

$$(2.3)\qquad \sum_{i=0}^{k} a_i W_{n-i,j} = \begin{cases} 0 \text{ if } j = 0(1)n - \bar{k} - 1 \\ b_{n-j} \text{ if } j = n - k\ (1)n \end{cases}, \ n \geq \bar{k}$$

where $a_i$ and $b_i$ are the coefficients of a k-step method for ODEs with characteristic polynomials

$$(2.4)\qquad \rho(z) = \sum_{i=0}^{\bar{k}} a_i z^{\bar{k}-i}, \quad \sigma(z) = \sum_{i=0}^{\bar{k}} b_i\ z^{\bar{k}-i}$$

Furthermore, we will use the forward shift operator E and we will occasionally write

$$\vec{F}_n = \vec{F}(t_n, y_n), \quad \vec{G}_n = \vec{G}(t_n).$$

THEOREM 2.1 *Let* K *be of the finitely decomposable form* (2.1) *and let the quadrature rule* (1.2b) *be* $(\rho,\sigma)$*-reducible with* $\rho(1) = 0$. *Then the VLM method can be convertad into the form*

$$\text{(2.5a)} \qquad \rho(E)\vec{U}_n = h\,\sigma(E)\vec{F}_n, \quad n \geq 0$$

$$\text{(2.5b)} \qquad \sum_{i=0}^{k} \alpha_i y_{n-i} = \sum_{i=0}^{k} \sum_{j=-k}^{k} [\gamma_{ij} h < \vec{G}_{n+j}, \vec{F}_{n-i} >$$

$$-\beta_{ij}(g(t_{n+j}) + < \vec{G}_{n+j}, \vec{U}_{n-i} >)], \quad n \geq k,$$

*where for* $j = 0(1)\ k-1$ *the vector* $\vec{u}_j$ *satisfies the starting condition*

$$g(t) + < \vec{G}(t), \quad \vec{U}_j > = Y_j(t). \qquad \square$$

PROOF. From the $(\rho,\sigma)$-reducibility property it follows that [15]

$$\rho(E)Y_n(t) = h\sigma(E)K_n(t), \quad n \geq 0.$$

Assuming that we can write

$$\text{(2.6)} \qquad Y_n(t) = g(t) + < \vec{G}(t), \quad \vec{U}_n >$$

for some vector $\vec{U}_n$, we obtain the equation

$$< \vec{G}(t), \quad \rho(E)\vec{U}_n - h\sigma(E)\vec{F}_n > = 0.$$

Apparently, by writing $Y_n(t)$ in the form (2.6) the vectors (2.5a) should satisfy (2.5a). The equation (2.5b) is immmediate by substituting (2.1) into (1.2a). Finally, the relation between the starting vectors $\vec{U}_j(t)$ follows from (2.6).

Notice that for decomposable kernels, the general VLM method requires only (n) kernel evaluations when written in the form (2.5).

In the following subsections two families of VLM methods will be discussed in view of the finite recurrence relation (2.5).

## 2.1 Direct quadrature methods

The conventional direct quadrature (DQ) method is obtained from (1.2) for k = 0, A = 1, B = -1 and C = 0. Substitution into theorem 2.1 immediately leads to the respresentation

$$(2.7) \qquad \rho(E)\vec{U}_n = h\sigma(E)\vec{F}(t_n,y_n), \qquad y_n = g(t_n) + \langle \vec{G}_n, \vec{U}_n \rangle .$$

Exactly the same relation is obtain if we apply the LM formula $\{\rho,\sigma\}$ to the system of ODEs (2.2). In other words, *for decomposable kernels, the DQ method is algebraically equivalent with an LM method applied to the system (2.2), provided that the starting values are also equivalent.* Hence, the numerical behaviour of the DQ method can be predicted on the basis of the well-developed theory for ODEs. Since the numerical behaviour of ODE solvers depend on the Jacobian matrix of the right side function of the system, we should consider the matrix

$$(2.8) \qquad J = (g_\nu(t)\frac{\partial f_\nu}{\partial y}(t,y(t))),$$

where $\mu$ and $\nu$ denote the row and column index, respectively. There are m-1 zero-eigenvalues and one eigenvalue given by

$$(2.9a) \qquad \lambda_m = \sum_{\mu=1}^{m} g_\mu(t)\,\frac{\partial f_\mu}{\partial y}(t,y) = \frac{\partial K}{\partial y}(t,t,y).$$

In most cases we can base the choice of the ODE solver on the spectrum of J that is if $\partial K/\partial y$ is of moderate size we choose an Adama-Moulton (AM) formula and if $\partial K/\partial y$ is e.g. large negative (indicating that the system (2.2) is *stiff*), we choose a backward differentation (BD) formula (see [11, p. 242]). The resulting quadrature formulas are respectively the Gregory rules (see e.g. [2, p. 117]) and the backward differentation quadrature rules. The latter, rather unconventional, rules were extensively discussed in [14]. In passing we remark that the BD quadrature rules are more expensive than the Gregory rules [15].

The selection of $\{\rho,\sigma\}$ on the basis of (2.9) is not safe in cases where the elements of J differ largely in magnitude. Such a situation is not reflected into the spectrum of J so that wrong conclusions might be drawn. However, by considering the Lipschitz constants of the succersive right-hand side functions in the systems (2.2) a safe criterion is obtained whether the system is stiff or nonstiff. Thus, instead of (2.9) we consider the Lipschitz conditions for the respective component equations in (2.2), that is

$$\| f_\mu(t,g(t)+<\vec{G}(t),\vec{U}(t)>) - f_\mu(t,g(t)+<\vec{G}(t),\vec{V}(t)>) \| \le L_\mu \| \vec{U}(t) - \vec{V}(t) \|,$$

where $\vec{V}$ lies in the neighbourhoud of the solution U and $L_\mu$ are the Lipschitz constants. With respect to the maximum norm we find

$$L_\mu \approx |g_\mu(t)| \sum_{\mu=1}^{m} |\frac{\partial f_\mu}{\partial y}(t,y(t))|. \quad (2.9b)$$

Thus, it is recommended to consider, in addition to (2.9a), the Lipschitz constants (2.9b)

## 2.2 Indirect linear multistep methods

The second method we investigated is the *indirect linear multistep* (ILM) method which was proposed in its general form in [10]. It arises if in (1.2a) $\alpha_i = a_i^*$, $\beta_{ij} = b_i^* d_{i+j}$, $\gamma_{ij} = b_i^* \delta_{i,j+2i}$ where $\{a_i^*, b_i^*\}$ define an LM formula $\{\rho^*,\sigma^*\}$, and $\{d_i\}$ are the coefficients of a forward differentiation formula, i.e.

$$hy'(t_n) \approx - \sum_{i=0}^{k} d_\ell Y(t_{n+\ell}) =: - \tau^*(E)Y(t_n).$$

THEOREM 2.2 *For finitely decomposable kernels of the form* (2.1) *the ILM method reduces to*

$$(2.10a) \qquad \rho(E)\vec{U}_n = h\sigma(E)\vec{F}(t_n,Y_n)$$

$$(2.10b) \qquad \rho^*(E)y_n = h\sigma^*(E)[K(t_n,t_n,y_n) - h^{-1}\tau^*(E)g(t_n) + < -h^{-1}\tau^*(E)\vec{G}_n, \vec{U}_n > ]. \qquad \square$$

PROOF. Substitution of the coefficients $\alpha_i$, $\beta_{ij}$ and $\gamma_{ij}$, defining the ILM method, into (2.5b) yields

$$\sum_{i=0}^{k} a_i^* y_{n-i} = h \sum_{i=0}^{k} b_i^*[< \vec{G}_{n-i}, \vec{F}_{n-i} > - h^{-1} \sum_{j=-i}^{k-i} d_{i+j}(g(t_{n+j}) + <\vec{G}_{n+j},\vec{U}_{n-i}>)]$$

$$= h \sum_{i=0}^{k} b_i^*[K(t_{n-i},t_{n-i},y_{n-i})-h^{-1} \sum_{j=0}^{k} d_j g(t_{n+j-i}) - h^{-1} < \sum_{j=0}^{k} d_j\vec{G}_{n+j-i},\vec{U}_{n-i} > ].$$

Writing this relation in terms of the polynomials $\rho^*,\sigma^*$ and $\tau^*$ we obtain (2.10b). Equation (2.10a) is immediate from (2.5a).

The recurrence relations (2.10) are recognized as *LM discretizations* of the system of ODEs

(2.13a) $\vec{U}'(t) = \vec{F}(t,y(t))$

(2.13b) $y'(t) = K(t,t,y(t)) + \langle \vec{G}'(t), \vec{U}(t) \rangle + g'(t).$

Hence, *for decomposable kernels, the ILM method is algebraically equivalent with LM methods* $\{\rho,\sigma\}$ and $\{\rho^*,\sigma^*\}$ *applied to the system* (2.13a) *and the equation* (2.13b), *provided that the starting values are also equivalent.*

Selecting suitable LM formulas $\{\rho,\sigma\}$ and $\{\rho^*,\sigma^*\}$, we obtain an ILM method which is expected to have a satisfactory numerical behaviour. The choice of suitable LM formulas depends on the Jacobian matrix

$$(2.14) \qquad J = \begin{pmatrix} 0 & \cdots & 0 & \partial f_1(t,y)/\partial_y \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ 0 & \cdots & 0 & \partial f_m(t,y)/\partial y \\ g'_1(t) & \cdots & g_m'(t) & \partial K(t,t,y)/\partial y \end{pmatrix}.$$

The eigenvalues of J are given by m-1 zero-eigenvalues and two eigenvalues $\lambda$ satisfying the equation

$$(2.15) \qquad \lambda^2 - \frac{\partial k}{\partial Y}(t,t,y)\lambda + \sum_{\mu=1}^{m} (-1)^{\mu} g_\mu'(t) \frac{\partial f_\mu}{\partial Y}(t,y) = 0.$$

In choosing the polynomials $\{\rho,\sigma\}$ and $\{\rho^*,\sigma^*\}$, we may exploit the fact that the system of equations in (2.13a) can be scaled in such a way that the $\partial f_\mu/\partial y$ are all small in magnitude, sothat (2.13a) presents an "innocent" system of ODEs. This is achieved by adapting the decomposition in (2.1). Of course, the functions $g_\mu(t)$, and consequently the derivatives $g_\mu'(t)$, may then be badly scaled, sothat the equation (2.13b) may become a difficult equation (for instance a stiff equation).

Systems where a subset of equations introduce time constants which differ by several orders of magnitude, whereas the other equations have a relatively slow behaviour, were investigated in e.g. [8] and [13 ]. These studies justify to integrate the innocent system (2.13a) by a *nonstiff* ODE solver $\{\rho,\sigma\}$, e.g. an AM-formula. Thus, we are led to an ILM $(\{\rho,\sigma\};\ \}\rho^*,\sigma^*\})$ method in which the lag terms is computed by Gregory rules and the LM formula is defined by polynomials $\{\rho^*,\sigma^*\}$ which correspond to a stiff solver if we have reasons to assume (2.13b) to be stiff (e.g. if $\partial K/\partial y$ is large negative), and which correspond to a nonstiff solver otherwise. Since the lagterm implementation is the more difficult one in the scheme (1.2), it is an attractive feature of the ILM method that we can use the simple and efficient Gregory rules for computing the lag term irrespective of the behaviour of the kernel K. This is a first advantage over the DQ method where in stiff cases the lag term should be computed by e.g. the backward differentiation quadrature rules which are rather difficult to implement.

It should be remarked that the systems of ODEs (2.2) and (2.13) may be largely different in characters. For instance, one system may be stiff and the other nonstiff. Furthermore, since the function $\vec{U}(t)$ does not necessarily have a physical interpretation, the systems of ODEs may be inherently unstable, whereas the VIE itself is perfectly stable. In case of (2.13), this difficulty can be overcome by choosing a $\{\rho^*,\sigma^*\}$ formula with a sufficiently large region of *relative stability*. In case of (2.2), however, $y(t)$ is not one of the unknown variables in the system, and is obtained by a weighted sum of the components of the unstable result $\vec{U}(t)$. If $y(t)$ is much smaller in magnitude than one or more of the components of $\vec{U}(t)$, we should expect large errors in applying the DQ method, irrespective of the lag term formula $\{\rho,\sigma\}$.

## 2.3 Other VLM methods

There have been proposed a few other families of linear multistep type methods which fit into the framework (1.2). In [16] so-called *multilag* (ML) and *modified multilag* (MML) methods were introduced.

These methods can be shown to be discretizations of the systems of ODEs (2.2) and (2.13), respectively. They are, however, *no* standard LM discretizations, such as the DQ and ILM methods, and we did not succeed in associating to them systems of ODEs for which the (M)ML methods can be interpreted as standard LM discretizations. Hence, we cannot just apply ODE theory to these methods for selecting suitable members of the (M)ML family.

## 3. A COMPARISON THEOREM

In the preceding section the kernel K was assumed of the finitely decomposable form (2.1). Here, we consider a general kernel K which is only assumed to be continuous in its arguments. Let $K^*$ be as decomposable approximation to K, then we want to know how the respective numerical solutions $y_n^*$ and $y_n$, produced by the same VLM method, differ from each other. We shall write $\varepsilon_n = y_n - y_n^*$.

<u>THEOREM 3.1</u> *Let the polynomial* $\alpha(z) = \quad \alpha_i z^{k-i}$ *satisfy the root condition (i.e. no roots outside the unit circle and those on the unit circle being simple). Let the row sums in the matrix* B *vanish, and let the following Lipschitz conditions be fulfilled in the domain of definition of* K *and* $K^*$:

$$|K(t,s,y) - K(t,s,\bar{y})| \le L_1 |y-\bar{y}|,$$

$$|K(t,s,y) - K(t,s,\bar{y}) - K(\bar{t},s,y) + K(\bar{t},s,\bar{y})| \le L_2 |t-\bar{t}|\,|y-\bar{y}|,$$

$$|K(t,s,y) - K^*(t,s,y)| \le \delta,$$

$$|K(t,s,y) - K^*(t,s,y) - K(\bar{t},s,y) + K^*(\bar{t},s,y)| \le \eta |t-\bar{t}|.$$

Then there exist a constant C independent of n and h such that for h sufficiently small

$$(3.1) \qquad \max_{k \le m \le N} |\varepsilon_n| \le C(\delta+\eta). \quad \square$$

PROOF. From (1.2) we derive by subtracting the two LM formulas

$$\sum_{i=0}^{k} \{\alpha_i \varepsilon_{n-i} + \sum_{j=-k}^{k} [\beta_{ij} \Delta Y_{n-i}(t_{n+j}) - \gamma_{ij} h(\Delta K_{n-i}(t_{n+j}) + \delta_{n-i}(t_{n+j}))]\} \tag{3.2}$$

where

$$\Delta Y_n(t) = Y_n(t) - Y_n^*(t) = h \sum_{\ell=0}^{n} w_{n\ell}[\Delta K_\ell(t) + \delta_\ell(t)]$$

$$\Delta K_\ell(t) = K(t,t_\ell,y_\ell) - K(t,t_\ell,y_\ell^*), \quad \delta(t) = K(t,t,y^*) - K^*(t,t_\ell,y_\ell^*)$$

We write (3.2) in the form

$$\alpha(E)\varepsilon_n = v_{n+k}, \quad n \geq 0. \tag{3.2'}$$

Since $\alpha(z)$ satisfies the root condition we have the inequality (see e.g. Stetter [14, p. 205]

$$|\varepsilon_n| \leq C_1 \sum_{j=k}^{n} |v_j|, \quad n \geq k, \quad C_1 \text{ constant} \tag{3.3}$$

where we have used that $\varepsilon_n = 0$ for $n < k$. For $v_n$ we derive the inequatility

$$\begin{aligned}
|v_n| \leq & \sum_{i=0}^{k} \{h \sum_{\ell=0}^{n-i} |w_{n-i,\ell}| |\sum_{j=-k}^{k} \beta_{ij}[\Delta K_\ell(t_{n+j}) + \delta_\ell(t_{n+j})]| \\
& + h \sum_{j=-k}^{k} |\gamma_{ij}[\Delta K_{n-i}(t_{n+j}) + \delta_{n-i}(t_{n+j})]|\} \\
\leq & C_2 \sum_{i=0}^{k} \{h \sum_{\ell=0}^{n} | \sum_{j=-k}^{k} [\beta_{ij}(|\Delta K_\ell(t_n) + \delta_\ell(t_n)| + |\Delta K_\ell(t_{n+j}) - \Delta K_\ell(t_n)| \\
& + |\delta_\ell(t_{n+j}) - \delta_\ell(t_n)|) + \gamma_{ij} h(|\Delta K_{n-i}(t_{n+j})| + |\delta_{n-i}(t_{n+j})|)]|\}
\end{aligned}$$

for some constant $C_2$. Using the Lipschitz conditions on K and $K^*$, and the conditions on $\beta_{ij}$ we find

$$|v_n| \leq C_3 \{h^2 \sum_{\ell=0}^{n} [|\varepsilon_\ell| + \eta] + h \sum_{i=0}^{k} |\varepsilon_{n-i}| + h\delta\}.$$

Substitution into (3.3) yields

$$|\varepsilon_n| \le C_4 \sum_{j=k}^{n} [h^2 \sum_{\ell=0}^{j} |\varepsilon_\ell| + h \sum_{i=0}^{k} |\varepsilon_{j-i}| + h^2(j+1)\eta + h\delta]$$

$$\le C_5 \{h(1+nh) \sum_{\ell=k}^{n} |\varepsilon_\ell| + n^2h^2\eta + nh\delta\}.$$

Finally, by applying a well-known Gronwall inequality (see e.g. [2, p. 925]) we find

$$\max_{k\le n\le N} |\varepsilon_n| \le C_5 \frac{nh(nh\eta+\delta)}{1-h(1+nh)C_5} \exp[C_5 \frac{nh(1+nh)}{1-h(1+nh)}],$$

where we assume h sufficiently small. Since $nh \le Nh = T - t_0$ the theorem is proved. □

According to the well-known theorem of Stone-Weierstrass, the class of continuous functions of the decomposable form (2.1.) is dense in the class of all continuous functions. Hence, K can be approximated by a decomposable function $K^*$ within any degree of accuracy (for an interesting discussion of this subject we refer to [6]). Consequently, the quantities $\delta$ and $\eta$ in the theorem can be made arbitrarily small. Thus, if we have a VLM method which satisfies the conditions of the theorem and if we know the numerical behaviour of that method for the class of decomposable kernels, then we know approximately its behaviour for non decomposable kernels. Such a method is the ILM method: it has an $\alpha(z)$ polynomial which satisfies the root condition because $\alpha(z) = \rho^*(z)$, the row sums of the matrix $B = (\beta_{ij})$ do vanish and, according to theorem 2.2, we can derive its numerical behaviour from known results for ODEs.

A similar theorem van be proved for the family of VLM methods for which $\alpha(z) = \alpha_0 z^k$ (without further conditions on the parameters $\beta_{ij}$ or $_{ij}$). Hence, in particular, the DQ method satisfies the relation (3.2) and therefore its numerical behaviour canbe predicted by using decomposable kernels as test kernels and by applying ODE theory to the system (2.2).

Finally, we remark that on the basis of this theorem one may decide to approximate the given kernels by a decomposable one and use the relations (2.5) directly for generating a numerical solution.

Such an approach has been advocated by Bownds in a number of papers [1,4-6] who starts with the system (2.2) and applies a suitable ODE solver (not necessarily an LM formula). Alternatively, one may start with the system (2.13) which is sometimes better conditioned (smaller Lipschitz constants) than the system (2.2) as we will see in the next section.

## 4. NUMERICAL ILLUSTRATION

Consider the following modification of an example given in [4]

$$J(t) = 50(1-t^2)\ell n(1+t) + 75t^2 - 51t + 1 - 100\int_0^t \ell n(1+t-s)y(s)ds, \quad 0 \le t \le T = 4, \tag{4.1}$$

with exact solution $y(t) = 1-t$. In order to illustrate the preceding discussion, we associate to the kernel in (4.1) the decomposable kernel

$$K^*(t,s,y) = \frac{20}{3}[t(t-10)y - 2t\,sy + s(10+s)y] \tag{4.2}$$

obtained by quadratic interpolation on the interval $0 \le t-s \le 4$, and define

$$\vec{F}(s,y) = [y, sy, s(10+s)y]^T$$

$$\vec{G}(t) = \frac{20}{3}[t(t-10), -2t, 1].$$

In case of the DQ method the Jacobian matrix J defined in (2.8) has m zero-eigenvalues. It is wrong, however, to conclude that the system (2.2) is an innocent system, because the Lipschitz constant L defined in (2.9b) differ largely in magnitude. For example, at t=4 we have $L_1 \cong 400$, $L_2 \cong 3200$ and $L_3 \cong 9600$. Consequently, we *should not expect the Gregory rules to be stable* in this problem, unless h is sufficiently small.

Next consider the ILM method with Gregory quadrature for the lag term. The choice of $\{\rho^*,\sigma^*\}$ depends on the matrix J defined in (2.14). The eigenvalues are given by (2.15) that is

$$\lambda_j(t) = 2\sqrt{\frac{10}{3}}\,(0,0,\sqrt{2t-5},-\sqrt{2t-5}).$$

Thus, for $0 \leq t < 5/2$ there are two purely imaginary eigenvalues the modulus of which decreases from ~ 8.17 until 0, and for $t > 5/2$ there is one positive and one negative eigenvalue with increasing modulus to become ~ 6.32 at $t = 4$. The moderate size of these eigenvalues indicate that it is not really crucial how $\{\rho^*,\sigma^*\}$ is chosen. The *AM formulas may be less accurate than expected for large h*, because of instability at the beginning and end of the integration interval, whereas the *higher order BD formulas may be less accurate for small h*, because of instability in the middle of the integration interval due to the interval of instability along the imaginary axis.

In table 4.1 the numbers of correct significant digits obtained at the point $t = 4$ are listed (this number is defined by $-\log_{10}$ (relative error)). The various methods are denoted by DQ ($G_r$), ILM ($G_r$ - $AM_p$) or ILM ($G_r$ - $BD_p$) where r denotes the order of the Gregory formula and p the order of the AM or BD formula. The results in table 4.1 confirm the predictions derived above).

Table 4.1 Results for problem (4.1)

| | h=1/4 | h=1/8 | h=1/16 | | h=1/4 | h=1/8 | h=1/16 |
|---|---|---|---|---|---|---|---|
| DQ ($G_2$) | * | * | 3.3 | DQ ($G_5$) | * | * | 2.3 |
| ILM ($G_2$-$AM_2$) | .6 | 1.1 | 1.2 | ILM ($G_5$-$AM_5$) | 1.1 | * | 4.3 |
| ILM ($G_2$-$BD_2$) | 1.8 | 2.2 | 2.7 | ILM ($G_5$-$BD_5$) | 2.5 | 2.2 | 2.6 |
| ILM ($G_2$-$BD_5$) | 1.5 | 3.6 | 2.3 | ILM ($G_5$-$BD_2$) | 1.0 | 2.0 | 2.5 |

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